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Bifurcation points of the generalized solution of the Hamilton-Jacobi-Bellman equation ⋆

A.S. Rodin * **L.G. Shagalova** **

* *N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of RAS, S. Kovalevskaya Str., 16, Yekaterinburg, 620990 Russia; Ural Federal University named after the first President of Russia B.N. Yeltsin (e-mail: alexey.rodin.ekb@gmail.com)*
** *N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of RAS, S. Kovalevskaya Str., 16, Yekaterinburg, 620990 Russia; Ural Federal University named after the first President of Russia B.N. Yeltsin (e-mail: shag@imm.uran.ru)*

Abstract: Properties of a minimax piecewise smooth solution of the Hamilton-Jacobi-Bellman equation are considered in the article. We study necessary and sufficient conditions for finding bifurcation points. Such points are points of "nucleation" of the set where the solution is not differentiable.

Keywords: Hamilton–Jacobi–Bellman equation, minimax solution, piecewise smooth solution, bifurcation points, singular set

1. INTRODUCTION

In this paper we study a piecewise-smooth minimax solution of the Hamilton-Jacobi-Bellman equation. Particular attention is paid to the problem of finding bifurcation points. One of the close works in this topic is Uspenskii and Lebedev (2018).

2. PIECEWISE SMOOTH SOLUTION OF HAMILTON–JACOBI–BELLMAN EQUATION

2.1 Problem statement

Consider the Cauchy boundary value problem for the Hamilton–Jacobi–Bellman equation

$$D_t\varphi(t,x) + H(t,x,D_x\varphi(t,x)) = 0, \quad \varphi(T,x) = \sigma(x), \quad (1)$$

where $t \in [0,T]$, $x \in R^n$,
and $D_x\varphi(t,x) = \Big(\dfrac{\partial\varphi(t,x)}{\partial x_1}, \dfrac{\partial\varphi(t,x)}{\partial x_2}, \dots, \dfrac{\partial\varphi(t,x)}{\partial x_n}\Big) = s.$

Define $\Pi_T = \{(t,x) : t \in [0,T], x \in R^n\}$, the symbol $int\Pi_T$ denotes the interior of the set Π_T.

We investigate problem (1) under the following assumptions:

(A1) the function $H(t,x,s)$ is continuously differentiable with respect to the variables t, x, s and it is concave(convex) with respect to the variable s;

(A2) the function $\sigma(x)$ is continuously differentiable;

(A3) there exist $\alpha > 0$ and $\beta > 0$ such that $\|D_xH(t,x,s)\| \le \alpha(1 + \|x\| + \|s\|)$,
$\|D_sH(t,x,s)\| \le \beta(1 + \|x\| + \|s\|)$
for any point $(t,x,s) \in \Pi_T \times R^n$.

Here, the symbol $\|\cdot\|$ denotes the Euclidean norm in R^n.

2.2 Generalized solution to problem (1)

Under the above assumptions, a classical solution $\varphi(\cdot)$ to problem (1) may exist only locally in a neighborhood of the boundary manifold

$$C^T = \{(t,x,z) : t = T, x = \xi, z = \sigma(\xi);\ \xi \in R^n\}.$$

This solution $\varphi(\cdot)$ can be constructed using the Cauchy method of characteristics Petrovskii (1984). The characteristic system with the boundary conditions at $t = T$ for problem (1) has the form:

$$\dot{\tilde{x}} = D_sH(t,\tilde{x},\tilde{s}), \quad \dot{\tilde{s}} = -D_xH(t,\tilde{x},\tilde{s}), \quad (2)$$

$$\begin{gathered} \dot{\tilde{z}} = \langle \tilde{s}, D_sH(t,\tilde{x},\tilde{s})\rangle - H(t,\tilde{x},\tilde{s}), \\ \tilde{x}(T,\xi) = \xi, \quad \tilde{s}(T,\xi) = D_x\sigma(\xi), \\ \tilde{z}(T,\xi) = \sigma(\xi) \quad \forall \xi \in R^n. \end{gathered} \quad (3)$$

The symbol $\langle\cdot,\cdot\rangle$ denotes the inner product.

The solutions $\tilde{x}$, $\tilde{s}$, and $\tilde{z}$ are called, respectively, the *state*, *adjoint*, and *cost characteristics* of the Hamilton–Jacobi–Bellman equation (1).

We note that, under conditions (A1)–(A3), for any $\xi \in R^n$ a solution of the characteristic system exists, it is unique, and can be extended to the interval $[0,T]$.

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According to the Cauchy method Petrovskii (1984), we have the formulas $x = \tilde{x}(t,\xi)$, $\varphi(t,x) = \tilde{z}(t,\xi)$, and $D_x\varphi(t,x) = \tilde{s}(t,\xi)$, $t \in [0,T]$, $\xi \in R^n$, if the Jacobian $\dfrac{\partial\tilde{x}(t,\xi)}{\partial(t,\xi)}$ is not equal to zero.

As a rule, classical solution does not exist in Π_T. So, we consider generalized solutions. We recall (see Subbotina et al. (2013),Kolpakova (2010)) the definition.

Definition 1. The superdifferential $D^+\varphi(t_0,x_0)$ of a function $\varphi(\cdot)\colon \Pi_T \to R$ at a point (t_0,x_0) is defined as the following set co $\Big\{(\alpha,s) \in R^{n+1} : \limsup\limits_{\Delta t\to 0,\|\Delta x\|\to 0}$

$$\frac{\varphi(t_0+\Delta t, x_0+\Delta x) - \varphi(t_0,x_0) - \langle(\alpha,s),(\Delta t,\Delta x)\rangle}{|\Delta t| + \|\Delta x\|} \le 0\Big\}.$$

The symbol coS denotes the convex hull of the set S.

Definition 2. A generalized solution to problem (1) is a locally Lipschitz superdifferentiable(subdifferentiable) function $\Pi_T \ni (t,x) \mapsto \varphi(t,x) \in R$ such that, for any point $(t_0,x_0) \in \Pi_T$, there exist $\xi_0 \in R^n$ and solutions $\tilde{x}(\cdot,\xi_0)$, $\tilde{s}(\cdot,\xi_0)$, and $\tilde{z}(\cdot,\xi_0)$ of system (2), (3) satisfying the condition

$$\tilde{x}(t_0,\xi_0) = x_0, \quad \tilde{z}(t_0,\xi_0) = \varphi(t_0,x_0),$$
$$\tilde{z}(t,\xi_0) = \varphi(t,\tilde{x}(t,\xi_0)) \quad \forall t \in [t_0,T].$$

The following assertion in valid on the connection of Definition 2 with the definitions of the minimax solution and the viscosity solution is a consequence of results in Subbotina et al. (2013), Kolpakova (2010), Subbotin (1995), Crandall and Lions (1983), Subbotina and Kolpakova (2010).

Proposition 3. If conditions (A1)–(A3) are satisfied to problem (1), then there exists a unique generalized solution to problem (1) in the sense of Definition 2. Definition 2 is equivalent to the definitions of the minimax solution and the viscosity solution to problem (1).

2.3 Singular set of a generalized solution

Let us recall the definition of the singular set of a generalized solution $\varphi(\cdot)$ to problem (1).

Definition 4. The singular set Q of a generalized solution $\varphi(\cdot)$ to problem (1) is the set of points $(t,x) \in \Pi_T$ where the function φ is not differentiable.

According to Subbotina et al. (2013), Kolpakova (2010), the following proposition holds.

Proposition 5. Let conditions (A1)–(A3) be satisfied for problem (1). Then $(t,x) \in Q$ if and only if there exist $\xi_1, \xi_2 \in R^n$, $\xi_1 \neq \xi_2$, such that $\tilde{x}(t,\xi_1) = \tilde{x}(t,\xi_2) = x$,

$$\tilde{z}(t,\xi_1) = \tilde{z}(t,\xi_2) = \varphi(t,x), \quad \tilde{s}(t,\xi_1) \neq \tilde{s}(t,\xi_2),$$

where $\tilde{x}(\cdot,\xi_i)$, $\tilde{s}(\cdot,\xi_i)$, and $\tilde{z}(\cdot,\xi_i)$, $i = 1,2$, are solutions of the characteristic system (2), (3).

2.4 Class of piecewise smooth functions

In this paper, we consider generalized solutions $\varphi(\cdot)$ to problem (1) in the class of piecewise smooth functions (see, for instance, Subbotin (1995)).

Definition 6. A continuous function $\varphi(\cdot) : \Pi_T \to R$ is called piecewise smooth in Π_T if

(1) the domain of this function Π_T has the following structure:

$$int\Pi_T = \bigcup_{i\in I} M_i, \quad M_i \cap M_j = \emptyset \quad \text{for} \quad i,j \in I,\ i \neq j,$$

where $I = \{1,2,\dots,N\}$, M_i are differentiable submanifolds in Π_T;

(2) the restriction of a piecewise smooth function $\varphi(\cdot)$ to $\overline{M}_j$, $j \in J$, is a continuously differentiable function, where $J := \{i \in I : M_i$ is an $(n+1)$-dimensional manifold$\}$, $\overline{M}_j$ is the closure of the set M_j;

(3) for any $i \in I$, $(t_1,x_1),(t_2,x_2) \in M_i$, the condition $J(t_1,x_1) = J(t_2,x_2)$ holds, where

$$J(t,x) := \{j \in J : x \in \overline{M}_j\}.$$

3. BIFURCATION POINTS

We introduce the following definition

Definition 7. The point of bifurcation (t^*,x^*) is the point $(t^*,x^*) \in \overline{Q} \setminus Q$.

We introduce the function

$$cross(\xi) = \min_{(t,\tilde{x}(t,\xi))\in\overline{Q}} (T-t),$$

where $cross(\cdot) : R^n \to [0;+\infty]$.

Remark 8. To every bifurcation point (t^*,x^*) there corresponds a point $(\xi^*, cross(\xi^*))$, which is a minimum point of the graph of the function $cross(\cdot)$.

3.1 Simple Hamiltonian

Consider the case as $H = H(s)$.

Theorem 9. Let conditions (A1)–(A3) be satisfied for problem (1), H=H(s), than the function $cross(\cdot)$ is continuous at the point ξ^*.

Proof. We prove the fact by reductio ad absurdum. Consider a point $(t,x) \in Q$ where at least two characteristics intersect that satisfy, and $(t^*,x^*) \in \overline{Q} \setminus Q$, the assertion (3).

Consider the difference for the state characteristic $\tilde{x}(\cdot)$:

$$x - x^* = \xi - \xi^* - (T-t)D_sH(D\sigma(\xi)) + (T-t^*)D_sH(D\sigma(\xi^*)).$$

From this equalities we can get

$$\|x - x^* + (t^* - t)D_sH(D\sigma(\xi))\| =$$
$$= \|\xi - \xi^* + (T-t^*)(D_sH(D\sigma(\xi^*) - D_sH(D\sigma(\xi))\|\,;$$

Let us pass to the limit, as $x \to x^*$, $t \to t^*$ and $\xi \to \xi^{**} \neq \xi^*$.

We get

$$0 = \|x - x^*\| + (t - t^*)\,\|D_sH(D\sigma(\xi))\| \ge$$
$$\ge \|\xi^{**} - \xi^* + (T-t^*)(D_sH(D\sigma(\xi^*) - D_sH(D\sigma(\xi^{**}))\|$$

Note, that

$$\lim_{\substack{x\to x^*\\ t\to t^*}} \|\xi^{**} - \xi^* + (T-t^*)(D_sH(s^*) - D_sH(s^*)\| = 0,$$

$s^* = D\sigma(\xi^*), \quad s^{**} = D\sigma(\xi^{**}).$

It means, that

$$\|\xi^{**} - (T - t^*) D_s H(D\sigma(\xi^{**})) - x^*\| = 0. \qquad (4)$$

Similarly, we obtain the condition for the cost characteristic.

Consider the difference for the cost characteristic:

$$\|\sigma(\xi^{**}) - (T - t^*)(\langle s^{**}, D_s H(s^{**})\rangle - H(s^{**})) - z^*\| = 0, (5)$$

where $s^{**} = D\sigma(\xi^{**})$.

It follows from conditions (4) and (5) that characteristics $(\tilde{x}(t^*, \xi^*), \tilde{z}(t^*, \xi^*), \tilde{s}(t^*, \xi^*))$ and $(\tilde{x}(t^*, \xi^{**}), \tilde{z}(t^*, \xi^{**}), \tilde{s}(t^*, \xi^{**}))$ are intersected at the point (t^*, x^*). Therefore $(t^*, x^*) \notin Q$. We have obtained a contradiction: the point $(t^*, x^*) \neq Q$.

Thus, we obtained, that $\xi^{**} = \xi^*$.

Theorem 10. Let conditions (A1)–(A3) be satisfied for problem (1), $H = H(s)$ and $D_{\xi_i}(D_{s_i} H(D\sigma(\xi)))$ is continuous. If there exists a bifurcation point (t^*, x^*), then the condition

$$D_{\xi_1}(D_{s_1} H(D\sigma(\xi^*))) = D_{\xi_i}(D_{s_i} H(D\sigma(\xi^*))), \quad i \in \overline{2, n},$$

is true.

Proof. We consider the condition when the state characteristic belongs to the singular set

$$\xi_1 - \xi_2 = (T - t)(D_s H(D\sigma(\xi_1)) - D_s H(D\sigma(\xi_2))),$$
$$T - t = cross(\xi_1) = cross(\xi_2).$$

We expand in a Taylor series the function $D_s H(D\sigma(\xi_1)) - D_s H(D\sigma(\xi_2))$ in the neighborhood of the point ξ^*:

$$\xi_1 - \xi_2 = cross(\xi_1)(D_\xi(D_s H(D\sigma(\xi^*))(\xi_1 - \xi^*)) -$$
$$-D_\xi D_s H(D\sigma(\xi^*))(\xi_2 - \xi^*) + M(\xi_1 - \xi_2)),$$

where $M, D_\xi(D_s H(D\sigma(\xi^*)) - n \times n -$ matrix, $(\xi_1 - \xi_2) \in R^n$.

Elements of matrix M have the form:
$M_{ij} = \Sigma_{k=1}^n(((\xi_1{}^k - \xi^{*k}) D_{\xi_j}(D_{\xi_k}(D_{s_i} H(D\sigma(\vartheta_{ikj}{}^1))))) + ((\xi_2{}^k - \xi^{*k}) D_{\xi_k}(D_{\xi_j}(D_{s_i} H(D\sigma(\vartheta_{ijk}{}^2)))))).$

Hence

$$E(\xi_1 - \xi_2) = cross(\xi_1)(D_\xi(D_s H(D\sigma(\xi^*))(\xi_1 - \xi_2)) + M(\xi_1 - \xi_2)).$$

We introduce a parametrization $\xi_1 = \xi_1(\tau)$, $\xi_2 = \xi_2(\tau)$, where $\tau = t - t^*$.

There exist a vector $h \neq 0$ and the smallest number r such that:

$$\xi_1(\tau) - \xi_2(\tau) = h(\tau)\tau^r,$$

where $h(\tau) \xrightarrow{\tau \to 0} h$.

We obtain in the limit $\tau \to 0$

$$Eh = cross(\xi^*)(D_\xi(D_s H(D\sigma(\xi^*)))h.$$
$$(E - cross(\xi^*)(D_\xi(D_s H(D\sigma(\xi^*)))))h = 0. \qquad (6)$$

Let us find a value ξ^* such that equality (6) holds for any h, then

$$E - cross(\xi^*)(D_\xi(D_s H(D\sigma(\xi^*)))) = 0,$$

or

$$\frac{1}{cross(\xi^*)} = D_{\xi_i}(D_{s_i} H(D\sigma(\xi^*))), \quad i \in \overline{1, n}.$$

Examples Let us illustrate the obtained conditions by examples.

Example 11. Let

$$H(s) = \sqrt{1 + s^2}, \quad \sigma(\xi) = sin(\xi), \quad \xi, s \in R.$$

We find $D_s H(s) = \frac{s}{\sqrt{1+s^2}}$,

$$D_{\xi_i} D_s H(\sigma(\xi)) = D_{\xi_i} \frac{cos(\xi)}{\sqrt{1 + cos^2(\xi)}} =$$
$$= \frac{-sin(\xi)}{(1 + cos^2(\xi))^{1,5}} \geq 0.$$

Consequently $\xi \in [-\pi + 2\pi k, 2\pi k], k \in Z$.

$cross(\xi) = -\frac{(1+cos^2(\xi))^{1,5}}{sin(\xi)}$.

In order to find a value ξ^*, it is necessary that $Dcross(\xi) = 0$:

$$Dcross(\xi) = 2\frac{(1 + cos^2(\xi))^{0,5} cos(\xi)(1 + sin^2(\xi))}{sin^2(\xi)} = 0.$$

Found $\xi^* = -\frac{\pi}{2} + 2\pi k, k \in Z$. Let us find the point of bifurcation (t^*, x^*):

$t^* = cross(\xi^*) = 1,$

$$x^* = \xi^* - (T - t^*) D_s H(D\sigma(\xi^*)) = -\frac{\pi}{2} + 2\pi k, \quad k \in Z.$$

$(t^*, x^*) = (1, -\frac{\pi}{2} + 2\pi k), k \in Z.$

Example 12.

$$H(s) = \sqrt{1 + \|s\|^2}, \quad \sigma(\xi) = \frac{\|\xi\|^2}{2}, \quad \xi, s \in R^n.$$

We find $D_{s_i} H(s) = \frac{s_i}{\sqrt{1+\|s\|^2}}$,

$$D_{\xi_i} D_{s_i} H(\sigma(\xi)) = D_{\xi_i}\left(\frac{\xi_i}{\sqrt{1 + \|\xi\|^2}}\right)$$

.

$$D_{\xi_i}\left(\frac{\xi_i}{\sqrt{1 + \|\xi\|^2}}\right) = \frac{\sqrt{1 + \|\xi\|^2} - \frac{\xi_i^2}{\sqrt{1+\|\xi\|^2}}}{1 + \|\xi\|^2}.$$

Consequently $\xi_i^2 = \xi_j^2 = a^2, \quad i, j \in \overline{1, n}$. We get

$$cross(a) = \frac{(1 + na^2)^{1,5}}{1 + (n - 1)a^2}.$$

In order to find a value $\xi^*(a)$, it is necessary that $Dcross(a) = 0$:

$$Dcross(\xi) = \frac{a((n + 2) + n(n - 1)a^2)}{(1 + (n - 1)a^2)} = 0.$$

Found $\xi^* = 0$. Let us find the point of bifurcation (t^*, x^*): $t^* = cross(\xi^*) = 1$, $x^* = \xi^* - (T - t^*) D_s H(D\sigma(\xi^*)) = 0$.

$(t^*, x^*) = (1, 0).$

3.2 Case when the Hamiltonian has the form $H = H(t, s)$

For the case of the Hamiltonian $H = H(t, s)$, the proof of the following theorems is carried out in a similar way as in the previous subsection.

Theorem 13. Let conditions (A1)–(A3) be satisfied for problem (1), $H = H(t, s)$, then the function $cross(\cdot)$ is continuous at the point ξ^*.

Theorem 14. Let conditions (A1)–(A3) be satisfied for problem (1), $H = H(t, s)$ and $D_{\xi_i}(D_{s_i}H(t, D\sigma(\xi)))$ is continuous with respect to the variables ξ. If there exists a bifurcation point (t^*, x^*), then the condition

$$1 = \int_{t^*}^{T} D_{\xi_1}(D_{s_1}H(\tau, D\sigma(\xi^*)))d\tau, \quad i \in \overline{1, n},$$

is true.

3.3 Case when the Hamiltonian has the form $H = H(x, s)$

In paper Subbotina and Shagalova (2017) the following Hamilton-Jacobi equation originated in molecular genetics for Crow–Kimura model of molecular evolution is studied.

$$D_t\varphi(t, x) + H(x, D_x\varphi(t, x)) = 0, \quad \varphi(0, x) = \sigma(x),$$

$x \in [-1, 1]$, $t \geq 0$,

where $H(x, s) = -f(x) + 1 - \frac{1+x}{2}e^{2s} - \frac{1-x}{2}e^{-2s}$,

and where the function $f(\cdot)$ called fitness is assumed to be twice continuously differentiable.

We write down the characteristic system for the Hamilton-Jacobi equation:

$$\dot{x} = -(1 + x)e^{2s} + (1 - x)e^{-2s};$$
$$\dot{s} = f^{'}(x) + \frac{e^{2s} - e^{-2s}}{2};$$
$$\dot{z} = sD_sH(x, s) - H(x, s),$$

with initial conditions:

$$x(0, \xi) = \xi, \quad s(0, \xi) = D\sigma(\xi), \quad z(0, \xi) = \sigma(\xi), \quad \xi \in [-1, 1].$$

Consider the system of the first two ODE.

$$\dot{x} = -(1 + x)e^{2s} + (1 - x)e^{-2s};$$
$$\dot{s} = f^{'}(x) + \frac{e^{2s} - e^{-2s}}{2}$$

We can reduce the system to an ordinary differential equation with respect to the phase characteristic:

$$\sqrt{\dot{x}^2 + 4(1 - x^2)} = 2(1 - H(\xi, D\sigma(\xi)) - f(x)), x \in (-1, 1).$$

In this case, if the condition

$$1 - H(\xi, D\sigma(\xi)) - f(x) > \sqrt{1 - x^2}$$

holds, then

$$\dot{x} = \pm 2\sqrt{x^2 - 1 + (1 - H(\xi, D\sigma(\xi)) - f(x))^2}$$

is true.

From here we can get the following connection

$$\left| \int_{\xi}^{x} \frac{dy}{\sqrt{y^2 - 1 + (1 - H(\xi, D\sigma(\xi)) - f(y))^2}} \right| = 2 \cdot \int_0^t dt.$$

Suppose $x^* > \xi_1 \geq \xi^* \geq \xi_2$, then we have the equalities

$$\int_{\xi_1}^{x} \frac{dy}{\sqrt{y^2 - 1 + (1 - H(\xi_1, D\sigma(\xi_1)) - f(y))^2}} = 2 \cdot \int_0^t dt;$$

$$2 \cdot \int_0^t dt = \int_{\xi_2}^{x} \frac{dy}{\sqrt{y^2 - 1 + (1 - H(\xi_2, D\sigma(\xi_2)) - f(y))^2}};$$

$$\int_{\xi^*}^{x^*} \frac{dy}{\sqrt{y^2 - 1 + (1 - H(\xi^*, D\sigma(\xi^*)) - f(y))^2}} = 2 \cdot \int_0^{t^*} dt;$$

Using the first two equations and passing to the limit, as $\xi_1 \to \xi^*$, $\xi_2 \to \xi^*$, we obtain

$$\frac{1}{\sqrt{\xi^{*2} - 1 + (1 - H(\xi^*, D\sigma(\xi^*)) - f(\xi^*))^2}} =$$
$$= \int_{\xi^*}^{x^*} \frac{(H(\xi^*, D\sigma(\xi^*)))'_{\xi}(1 - H(\xi^*, D\sigma(\xi^*)) - f(y))dy}{(y^2 - 1 + (1 - H(\xi^*, D\sigma(\xi^*)) - f(y))^2)^{1,5}}.$$

A necessary conditions for finding the bifurcation point are

$$\int_{\xi^*}^{x^*} \frac{dy}{\sqrt{y^2 - 1 + (1 - H(\xi^*, D\sigma(\xi^*)) - f(y))^2}} = 2t^*;$$

$$\frac{1}{\sqrt{\xi^{*2} - 1 + (1 - H(\xi^*, D\sigma(\xi^*)) - f(\xi^*))^2}} =$$
$$= \int_{\xi^*}^{x^*} \frac{(H(\xi^*, D\sigma(\xi^*)))'_{\xi}(1 - H(\xi^*, D\sigma(\xi^*)) - f(y))dy}{(y^2 - 1 + (1 - H(\xi^*, D\sigma(\xi^*)) - f(y))^2)^{1,5}}.$$

4. CONCLUSION

In the paper, for different types of Hamiltonians, it was possible to obtain the necessary conditions for finding the bifurcation point. The results are illustrated by examples.

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